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The Embeddability of Affine *PI*-Algebras in Rings of Matrices

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In this paper we examine the problem of whether a *PI*-algebra E, affine over a field K and satisfying all identities of $M_n(K)$ for some n, can be embedded in $M_r(C)$ for some positive integer r and commutative K-algebra C. For brevity, we will say E is embeddable in matrices if such an r and C exist. Among other results, we prove there is an E not embeddable in matrices, although all of its proper chains of left or right annihilator ideals have length at most 4, answering a question in [4]. Let us review earlier work on the problem, describing our results along the way.

Any subalgebra of a matrix ring over a commutative K-algebra is a *PI*-algebra, but the converse fails, as illustrated by the exterior algebra of an infinite-dimensional vector space over a characteristic 0 field, which satisfies no standard identity. Amitsur constructed the first example of a *PI*-algebra satisfying all $n \times n$-matrix identities which is not embeddable in matrices [2]. The nonembeddability is a consequence of the main theorem in Amitsur's paper (although he sidesteps the theorem in proving non-embeddability, relying instead on direct calculation); the theorem states: any chain $I_0 \subseteq I_1 \subseteq \cdots$, of ideals in an affine algebra E such that E/I_j embeds in $M_{r_j}(C_j)$, with a bound on the r_j's, must stabilize. Thus, if E has a proper infinite chain of ideals $\{I_j\}$ and all the algebras E/I_j are embeddable in matrices, the sum of the E/I_j's yields a nonembeddable example, as in

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Amitsur's paper. The resulting example is not affine, however, and Amitsur observed that any affine subalgebra of $M_r(C)$ would have a nilpotent Jacobson radical, leading him to ask the following question: if an affine algebra satisfies all identities of $n \times n$ matrices and has a nilpotent Jacobson radical, is it embeddable in matrices? (Later work of Lewin, Kemer, and Braun has shown that these two assumptions are equivalent and are satisfied by any affine PI-algebra [3, 5, 6].)

An example of Small answered the question negatively [11]. Small's example fails to satisfy the ascending chain condition on left or right annihilator ideals. But, as observed in [11], if an affine algebra E embeds in an $M_r(C)$, it also embeds in $M_r(C')$ for some affine K-subalgebra C' of C. Thus E inherits a.c.c. on left and right annihilator ideals from the noetherian algebra $M_r(C')$, and this is how one concludes that Small's example is not embeddable in matrices.

Lewin showed that a negative answer can also be obtained by a simple counting argument [7]. Given an affine algebra E over a countable field K, satisfying Amitsur's two conditions, and an uncountable family $\{I_\alpha\}$ of distinct ideals of E, some E/I_α is not embeddable in matrices. For, the preceding observation implies that there are only countably many isomorphism classes of algebras affine over K and embeddable in matrices. But by the countability of K, there must be uncountably many distinct isomorphism classes of algebras in the set $\{E/I_\alpha\}$: a fixed affine algebra over a countable field can have only countably many presentations.

In [9], following an account of Small's example, Procesi asked if a.c.c. on left and right annihilator ideals, along with Amitsur's two conditions, would imply embeddability in matrices. An example of Irving answered this question negatively [4]; this example satisfies a.c.c. on annihilator ideals, but there are proper chains of arbitrary length. Nonembeddability follows because a commutative noetherian ring C' can be embedded in a commutative artinian ring C'' [10], so any affine subalgebra of $M_r(C')$ inherits a bound on the length of proper chains of annihilator left or right annihilator ideals from $M_r(C'')$. This suggests a reformulation of Procesi's question, as noted in [4]: do Amitsur's two conditions, along with boundedness of the length of proper chains of left and right annihilator ideals, imply embeddability in matrices? In Section 2 this question is answered negatively. Let $A = K\{x, y\}/(x^2, yxy)$ with K a countable field, and let α be a subset of $\mathbb{N}^+$. It is evident that A has Jacobson radical (x) with $(x)^3 = (0)$, and we will see in Section 3 that A embeds in $M_4(K[u])$, so any homomorphic image satisfies Amitsur's two conditions. Each element xy^ix of A spans an ideal, so the set $\{xy^ix \mid i \in \alpha\}$ spans an ideal I_α. Let $A_\alpha = A/I_\alpha$. The example in [4] was of the form A_α for suitable choice of α. In Section 2 we prove that A_α has no proper chain of left or right annihilator ideals of length greater than 4, if α is a set $\{i_1, i_2, \dots\}$ satisfying

$i_{n+1} > 2i_n$ for all $n \in \mathbb{N}^+$. There are uncountably many such sets: for instance, for each real number $s \geqslant 3$, let α consist of the numbers $i_n = [s^n]$. Therefore, we may conclude by Lewin's argument that one such A_α is not embeddable in matrices, yielding the negative answer.

This would seem to dash the hope that affine subalgebras of matrix rings can be characterized by Amitsur's conditions and additional chain conditions. There is a different sort of characterization available, due to Mal'cev [8]: an affine algebra E is embeddable in matrices if and only if E is a subdirect product of finite-dimensional K-algebras whose dimensions are uniformly bounded. Conditions which are sufficient for embeddability in matrices would still be desirable, and the obvious question outstanding is whether affine, noetherian PI-algebras are embeddable in matrices. As a partial result, Small has proved that an algebra E which is a finite module over its center Z, with Z noetherian and containing a field, is embeddable in matrices [12]. Further positive evidence is provided in Section 4 by a result obtained jointly with Amitsur: there are only countably many isomorphism classes of affine, noetherian PI-algebras over a fixed countable field. Thus, no nonembeddable affine, noetherian PI-algebra can be produced via Lewin's argument.

In Section 3 we address a question untouched by Amitsur's theorem mentioned earlier. Given an affine algebra E and a chain of ideals $I_0 \subseteq I_1 \subseteq \cdots$, such that each E/I_j embeds in some $M_{r_j}(C_j)$, with no bound assumed on the r_j's, must the chain stabilize? We prove that the algebra A_α defined earlier embeds in some $M_r(C)$ if α is a finite or cofinite subset of $\mathbb{N}^+$. In particular, a negative answer to the question is provided by the chain of ideals $I_{\alpha_1} \subset I_{\alpha_2} \subset \cdots$ in A corresponding to the subsets $\alpha_n = \{1,..., n\}$. There are embeddings $A_{\alpha_n} \to A_{\alpha_{sn}}$ for each s and n, and the direct limit of these inclusions is an algebra A_∞ satisfying the identities of A but not embeddable in matrices: if A_∞ were embeddable in some $M_r(C)$, each A_{α_n} would be as well, contradicting Amitsur's theorem.

All our examples have companions arising by replacing A with certain other algebras containing infinitely many independent one-dimensional ideals. In particular, A can be replaced by $K\{x, y, z\}/(zy, yx, zx, x^2, z^2)$, with ideals (xy^iz). This choice has the advantage of being embeddable in $M_3(K[t, u])$, so it satisfies all 3×3 matrix identities. Another substitute for A is the algebra

$$\begin{pmatrix} K & 0 & K[t] & K[t] \\ 0 & K & 0 & K \\ 0 & 0 & K[t] & K[t] \\ 0 & 0 & 0 & K \end{pmatrix}$$

with ideals spanned by t^ie_{14}. A close relative of this is the source of the first affine algebra not embeddable in matrices [11], and one could regard all the examples of this paper (as well as the first example of [4], in retrospect) as descendants.

Additional affine algebras not embeddable in matrices may be found in [12] and [4]. In particular, there are examples which satisfy a.c.c. on two-sided ideals and are subdirectly irreducible, in contrast to the examples arising as homomorphic images of A.

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Let K be a field and let $A = K\{x, y\}/(x^2, yxy)$, as in Section 1. Let α be a subset $\{i_1, i_2, ...\}$ of $\mathbb{N}^+$ and let I_α be the ideal spanned by $(xy^ix \mid i \in \alpha\}$. Let $B = A/I_\alpha$.

THEOREM. *Assume $i_{n+1} > 2i_n$ for all $n \in \mathbb{N}^+$. Let I be a left annihilator ideal and J a right annihilator ideal of B, with $I = l.\ \mathrm{ann}(J)$ and $J = r\ \mathrm{ann}(I)$. One of the following holds:*

(i) $I = Bx + By$, $J = xyB$,

(ii) $I = Byx$, $J = xB + yB$,

(iii) $I = Byx + xU$, $J = Vx + xyB$, *with U and V subspaces of $K[y]$ such that $xUVx = (0)$. In case* (iii), *either U or V is one dimensional.*

COROLLARY 1. *Given B as in the theorem, any chain of left or right annihilator ideals has at most four distinct elements.*

COROLLARY 2. *There exists an affine K-algebra satisfying the identities of $M_4(K)$, with Jacobson radical J satisfying $J^3 = (0)$, and with no proper chain of left or right annihilator ideals of length greater than 4, which is not embeddable in $M_r(C)$ for any commutative K-algebra C and positive integer r.*

Proof. For a countable field over K, the argument follows as described in Section 1, once we know that A embeds in $M_4(K[u])$. This will be proved in Section 4. For arbitrary K, the nonembeddability is obviously preserved under field extension. ∎

Remark. It is easy to show directly, for distinct subsets α, β of $\mathbb{N}^+$, that A/I_α and A/I_β are not isomorphic. This eliminates the need for one of the counting arguments indicated in Section 1 for the preceding proof. It would of course be preferable to eliminate the counting arguments entirely, proving directly that A/I_α is not embeddable in matrices for one (or all) of the subsets α satisfying the hypothesis of the theorem.

Proof of the Theorem. Lemma 1 of [4] shows that the left and right annihilator ideals of B have the form indicated in (i)–(iii), so only the one dimensionality of U or V requires proof. Two observations are required:

(a) Given I and J as in (iii), one of the spaces U or V consists of polynomials all of the same degree.

The proof of this is a simplification of the proof of Lemma 2 in [4]. Let $p_1(y)$, $p_2(y)$ be polynomials in U with degrees a_1 and a_2, and assume $a_1 < a_2$. Let $q_1(y)$, $q_2(y)$ be polynomials in V with degrees b_1 and b_2, and suppose $b_1 \leqslant b_2$. The product $xp_i(y)\,q_j(y)x = 0$ for $1 \leqslant i, j \leqslant 2$, which implies that the highest degree monomial $xy^{a_i+b_j}x$ in the product is 0, forcing $a_i + b_j$ to lie in α. The hypothesis on α implies that if m_1, m_2, m_3, and m_4 are elements of α with $m_1 - m_2 = m_3 - m_4$, then $m_1 = m_3$ and $m_2 = m_4$. But $(a_2 + b_2) - (a_1 + b_2) = (a_2 + b_1) - (a_1 + b_1)$. Hence $b_1 = b_2$, proving (a).

(b) If $p(y)$ and $q(y)$ are monic polynomials in $K[y]$ with $\deg p(y) \geqslant \deg q(y)$ and $xp(y)\,q(y)x = 0$ in B, then $q(y)$ is uniquely determined by its degree.

Let $p(y) = \sum_{i=0}^{m} r_i y^i$ and $q(y) = \sum_{j=0}^{n} s_j y^j$ with $r_m = s_n = 1$, and view the r_i's as fixed scalars of K, with the s_j's as unknowns for $0 \leqslant j \leqslant n-1$. The condition $xp(y)\,q(y)x = 0$ yields a family of equations

$$(r_m s_{n-d} + r_{m+1} s_{n-d-1} + \cdots + r_{m-d} s_n)\, xy^{m+n-d}x = 0$$

for $0 \leqslant d \leqslant m+n$. The condition $r_m s_n xy^{m+n}x = 0$ means that $m+n \in \alpha$. Let t be the largest element in α less than $m+n$. By assumption, $m \geqslant n$ and $m+n > 2t$, so $t < m$. Hence,

$$r_m s_{n-d} + \cdots + r_{m-d} s_n = 0$$

for $1 \leqslant d \leqslant n$, since for such choice of d, the monomial $xy^{m+n-d}x$ is not 0 in B. This makes it possible to obtain unique solutions, inductively, for s_{n-1}, $s_{n-2}, \ldots, s_0$, proving (b).

The theorem can now be easily proved. Let I and J be chosen as in (iii), and suppose the polynomials in U all have degree m. If U has distinct monic polynomials $p_1(y)$ and $p_2(y)$, then a dual version of (b) implies that the polynomials in V must all have degree $< m$. The degree n of a polynomial in V satisfies $m+n \in \alpha$. But there can be at most one positive integer n satisfying $n < m$ and $m+n \in \alpha$, for there is at most one integer in α

between m and $2m$. Therefore the degree n of a polynomial in V is uniquely determined. Applying (b) again, we conclude that V has a unique monic polynomial, so that V is one dimensional. This proves, if all polynomials in U have the same degree, that U or V is one dimensional. The symmetric argument works if all polynomials in V have the same degree. Hence, by (a), the theorem is proved. ▮

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Let A continue to denote the algebra $K\{x, y\}/(x^2, yxy)$ over a field K, and let α be a finite subset of $\mathbb{N}^+$. Let I_α be defined as in Section 1.

PROPOSITION. *Assume the largest integer in α is n and $\{1,..., n\} - \alpha = \{i_1,..., i_m\}$ with $i_1 < \cdots < i_m$. Then A/I_α embeds in $M_{n+4}(K[t, u])$ with t and u algebraically independent over K.*

Proof. Let $v_{-1}, v_0,..., v_{n+2}$ be a basis for the free $K[t, u]$-module of rank $n+4$ and let $\bar{x}, \bar{y}$ be elements of $M_{n+4}(K[t, u])$ defined by the following operations:

$$\begin{array}{llll}
v_{-1}\bar{x} = v_0, & & v_{-1}\bar{y} = 0, & \\
v_0\bar{x} = 0, & & v_i\bar{y} = v_{i+1}, & \text{if } 0 \leqslant i \leqslant n. \\
v_i\bar{x} = 0, & \text{if } i \in \alpha, & v_{n+1}\bar{y} = uv_{n+1}, & \\
v_{i_j}\bar{x} = t^j v_{n+2}, & & v_{n+2}\bar{y} = 0, & \\
v_{n+1}\bar{x} = v_{n+2}, & & & \\
v_{n+2}\bar{x} = 0, & & &
\end{array}$$

It is easily checked that $\bar{x}^2 = 0$, $\bar{y}\bar{x}\bar{y} = 0$, and $\bar{x}\bar{y}^i\bar{x} = 0$ for $i \in \alpha$. Thus we obtain a K-algebra homomorphism of A/I_α into $M_{n+4}(K[t, u])$ by sending x to $\bar{x}$ and y to $\bar{y}$. The injectivity of this map follows from the K-linear independence of the set $\{\bar{x}\bar{y}^i, \bar{y}^{i+1}\bar{x}, \bar{y}^i \mid i \in \mathbb{N}\} \cup \{\bar{x}\bar{y}^i\bar{x} \mid i \in \mathbb{N}^+ - \alpha\}$. ▮

Remarks. (1) If α is empty, then we may take $m = n = 0$ and obtain the desired embedding of A into $M_4(K[u])$.

(2) Let $A_0 = A$ and for $n \in \mathbb{N}^+$, let $A_n = A/I_\alpha$ with $\alpha = \{1,..., n\}$. The sequence of surjective homomorphisms $A_0 \to A_1 \to \cdots$, yields a proper chain of ideals $(0) = I_0 \subset I_1 \subset \cdots$, in A with each A/I_n embeddable in $M_{n+4}(K[u])$. As observed in Section 1, this demonstrates that Amitsur's theorem cannot be extended. Moreover, Amitsur's theorem may be applied to deduce that the size of matrices required for embedding the algebras A_n must become arbitrarily large as n increases. One would expect that the

size of matrices required for embedding A_{n+1} is larger than that for A_n. For each pair of positive integers s and n, there is a natural embedding of A_n in A_{sn} defined by sending the generators x and y of A_n to ξ and η^s in A_{sn}, where ξ and η are the generators of A_{sn}. The direct limit A_∞ of these embeddings is not embeddable in matrices, by Amitsur's theorem, but every affine subalgebra is. In contrast, the direct limit of the surjections $A_0 \to A_1 \to \cdots$, can be embedded in matrices. It is the algebra A_α with $\alpha = \mathbb{N}^+$, and its embeddability is a consequence of the next result.

PROPOSITION. *Let α be a cofinite subset of $\mathbb{N}^+$ and let $\mathbb{N}^+ - \alpha = \{i_1, ..., i_m\}$ with largest element n. Then A/I_α embeds in $M_{n+5}(K[t])$.*

Proof. Let $v_{-1}, v_0, ..., v_{n+3}$ be a basis for the free $K[t]$-module of rank $n+5$ and let $\bar{x}$, $\bar{y}$ be elements of $M_{n+5}(K[t])$ defined by the following operations:

$$
\begin{aligned}
&v_{-1}\bar{x} = v_0, && v_{-1}\bar{y} = 0,\\
&v_0\bar{x} = 0, && v_i\bar{y} = v_{i+1} \quad \text{if } 0 \leqslant i \leqslant n,\\
&v_i\bar{x} = 0 \quad \text{if } i \in \alpha \text{ and } i \leqslant n, && v_{n+1}\bar{y} = tv_{n+1},\ v_{n+2}\bar{y} = tv_{n+2},\\
&v_{i_j}\bar{x} = t^j v_{n+3}, && v_{n+3}\bar{y} = 0.\\
&v_{n+1}\bar{x} = v_{n+3}\bar{x} = 0, &&\\
&v_{n+2}\bar{x} = v_{n+3}, &&
\end{aligned}
$$

Then the map of A to $M_{n+5}(K[t])$ in which x is sent to $\bar{x}$ and y to $\bar{y}$ defines an injective algebra homomorphism of A/I_α into $M_{n+5}(K[t])$. ∎

Remark. In case $\alpha = \mathbb{N}^+$, one can set $n = -1$ in the proposition, obtaining an embedding of A/I_α in $M_4(K[t])$.

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THEOREM. *Let R be an affine, noetherian, semiprime PI-algebra over a countable field K. There are only countably many isomorphism classes of affine, noetherian K-algebras with R as image modulo the nilradical.*

Proof. The nilradical of a noetherian algebra is nilpotent, so it suffices to prove that there are countably many isomorphism classes of affine noetherian algebras with nilradical of fixed index of nilpotence and R as image modulo the nilradical. We may proceed by induction on the index, the case of index 1 yielding only the algebra R itself.

Suppose the theorem has been proved for algebras with index of nilpotence $\leqslant k$ and let E be an affine, noetherian algebra with nilradical N such that $N^{k+1}=(0)$ and $E/N \cong R$. It follows from a theorem of Lewin [6] that E embeds in the algebra

$$\begin{pmatrix} R & T \\ 0 & E/N^k \end{pmatrix}$$

for a particular $R-E/N^k$ bimodule T, generated as bimodule by a set of cardinality equal to the number of algebra generators of E. Thus, every affine, noetherian algebra E with index of nilpotence $k+1$ and image R modulo the radical embeds in an algebra of the form

$$\begin{pmatrix} R & T \\ 0 & S \end{pmatrix},$$

where S is an affine noetherian algebra of index of nilpotence $\leqslant k$ with image R modulo the radical, and T is a finitely-generated $R-S$ bimodule. By induction, there are only countable many choices for S, up to isomorphism, and the choices for T are exhausted by all homomorphic images of the free $R-S$ bimodules $(R \otimes_K S)^n$, for $n \in \mathbb{N}^+$. Since R and S are noetherian, there are only countable many subbimodules of $(R \otimes_K S)^n$ [12], and therefore only countably many choices of T. Hence E is an affine subalgebra of one of only countably many algebras of the form $\left(\begin{smallmatrix} R & T \\ 0 & S \end{smallmatrix}\right)$, so that there are only countably many choices for E, up to isomorphism. ▌

COROLLARY. *Let K be a countable field. There are only countable many isomorphism classes of noetherian PI-algebras affine over K.*

Proof. By the theorem, it suffices to pove that there are only countably many isomorphism classes of semiprime noetherian PI-algebras affine over K. But a theorem of Amitsur states that any semiprime PI-algebra is embeddable in matrices [1]. Hence there are only countably many isomorphism classes of affine, semiprime PI-algebras, noetherian or not. ▌

It seems reasonable to ask if there are only countably many isomorphism classes of affine noetherian algebras (not necessarily PI) over a fixed countable field K. There seems to be known example of an affine, noetherian algebra which is not finite-presented, and if all are, a positive answer would immediately follow.

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